

TAYLOR'S FORMULA AND INTEGRAL INEQUALITIES FOR CONFORMABLE FRACTIONAL DERIVATIVES

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ABSTRACT. We derive Taylor's Formula for conformable fractional derivatives. This is then employed to extend some recent and classical integral inequalities to the conformable fractional calculus, including the inequalities of Steffensen, Čebyšev, Hermite-Hadamard, Ostrowski, and Grüss.

1. TAYLOR THEOREM

We use the conformable α -fractional derivative, recently introduced in [4, 7], which for $\alpha \in (0, 1]$ is given by

$$(1.1) \quad D_* f(t) := \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \quad D_* f(0) = \lim_{t \rightarrow 0^+} D_* f(t).$$

Note that if f is differentiable, then

$$(1.2) \quad D_\alpha f(t) = t^{1-\alpha} f'(t),$$

where $f'(t) = \lim_{\varepsilon \rightarrow 0} [f(t + \varepsilon) - f(t)]/\varepsilon$.

We will consider Taylor's Theorem in the context of iterated fractional differential equations. In this setting, the theorem will be proven using the variation of constants formula, where we use an approach similar to that used for integer-order derivatives found in [6]. With this in mind, we begin this note with a general higher-order equation. For $n \in \mathbb{N}_0$ and continuous functions $p_i : [0, \infty) \rightarrow \mathbb{R}$, $1 \leq i \leq n$, we consider the higher-order linear α -fractional differential equation

$$(1.3) \quad Ly = 0, \quad \text{where} \quad Ly = D_\alpha^n y + \sum_{i=1}^n p_i D_\alpha^{n-i} y,$$

where $D_\alpha^n y = D_\alpha^{n-1}(D_\alpha y)$. A function $y : [0, \infty) \rightarrow \mathbb{R}$ is a solution of equation (1.3) on $[0, \infty)$ provided y is n times α -fractional differentiable on $[0, \infty)$ and satisfies $Ly(t) = 0$ for all $t \in [0, \infty)$. It follows that $D_\alpha^n y$ is a continuous function on $[0, \infty)$.

2010 *Mathematics Subject Classification.* 26A33.

Key words and phrases. Cauchy Function, Variation of Constants, Taylor's Formula.

Now let $f : [0, \infty) \rightarrow \mathbb{R}$ be continuous and consider the non-homogeneous equation

$$(1.4) \quad D_\alpha^n(t) + \sum_{i=1}^n p_i(t) D_\alpha^{n-i} y(t) = f(t).$$

Definition 1.1. We define the Cauchy function $y : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ for the linear fractional equation (1.3) to be, for each fixed $s \in [0, \infty)$, the solution of the initial value problem

$$Ly = 0, \quad D_\alpha^i y(s, s) = 0, \quad 0 \leq i \leq n-2, \quad D_\alpha^{n-1} y(s, s) = 1.$$

Remark 1.1. Note that

$$y(t, s) := \frac{1}{(n-1)!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^{n-1}$$

is the Cauchy function for $D_\alpha^n = 0$, which can be easily verified using (1.2).

Theorem 1.1 (Variation of Constants). *Let $\alpha \in (0, 1]$ and $s, t \in [0, \infty)$. If f is continuous, then the solution of the initial value problem*

$$Ly = f(t), \quad D_\alpha^i y(s) = 0, \quad 0 \leq i \leq n-1$$

is given by

$$y(t) = \int_s^t y(t, \tau) f(\tau) \tau^{\alpha-1} d\tau,$$

where $y(t, \tau)$ is the Cauchy function for (1.3).

Proof. With y defined as above and by the properties of the Cauchy function we have

$$D_\alpha^i y(t) = \int_s^t D_\alpha^i y(t, \tau) f(\tau) \tau^{\alpha-1} d\tau + D_\alpha^{i-1} y(t, t) f(t) = \int_s^t D_\alpha^i y(t, \tau) f(\tau) \tau^{\alpha-1} d\tau$$

for $0 \leq i \leq n-1$, and

$$\begin{aligned} D_\alpha^n y(t) &= \int_s^t D_\alpha^n y(t, \tau) f(\tau) \tau^{\alpha-1} d\tau + D_\alpha^{n-1} y(t, t) f(t) \\ &= \int_s^t D_\alpha^n y(t, \tau) f(\tau) \tau^{\alpha-1} d\tau + f(t). \end{aligned}$$

It follows from these equations that

$$D_\alpha^i y(s) = 0, \quad 0 \leq i \leq n-1$$

and

$$Ly(t) = \int_s^t Ly(t, \tau) f(\tau) \tau^{\alpha-1} d\tau + f(t) = f(t),$$

and the proof is complete. \square

Theorem 1.2 (Taylor Formula). *Let $\alpha \in (0, 1]$ and $n \in \mathbb{N}$. Suppose f is $(n + 1)$ times α -fractional differentiable on $[0, \infty)$, and $s, t \in [0, \infty)$. Then we have*

$$f(t) = \sum_{k=0}^n \frac{1}{k!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^k D_\alpha^k f(s) + \frac{1}{n!} \int_s^t \left(\frac{t^\alpha - \tau^\alpha}{\alpha} \right)^n D_\alpha^{n+1} f(\tau) \tau^{\alpha-1} d\tau$$

Proof. Let $g(t) := D_\alpha^{n+1} f(t)$. Then f solves the initial value problem

$$D_\alpha^{n+1} x = g, \quad D_\alpha^k x(s) = D_\alpha^k f(s), \quad 0 \leq k \leq n.$$

Note that the Cauchy function for $D_\alpha^{n+1} y = 0$ is

$$y(t, s) = \frac{1}{n!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^n.$$

By the variation of constants formula,

$$f(t) = u(t) + \frac{1}{n!} \int_s^t \left(\frac{t^\alpha - \tau^\alpha}{\alpha} \right)^n g(\tau) \tau^{\alpha-1} d\tau,$$

where u solves the initial value problem

$$(1.5) \quad D_\alpha^{n+1} u = 0, \quad D_\alpha^m u(s) = D_\alpha^m f(s), \quad 0 \leq m \leq n.$$

To validate the claim that $u(t) = \sum_{k=0}^n \frac{1}{k!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^k D_\alpha^k f(s)$, set

$$w(t) := \sum_{k=0}^n \frac{1}{k!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^k D_\alpha^k f(s).$$

Then $D_\alpha^{n+1} w = 0$, and we have that

$$D_\alpha^m w(t) = \sum_{k=m}^n \frac{1}{(k-m)!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^{k-m} D_\alpha^k f(s).$$

It follows that

$$D_\alpha^m w(s) = \sum_{k=m}^n \frac{1}{(k-m)!} \left(\frac{s^\alpha - s^\alpha}{\alpha} \right)^{k-m} D_\alpha^k f(s) = D_\alpha^m f(s)$$

for $0 \leq m \leq n$. We consequently have that w also solves (1.5), and thus $u \equiv w$ by uniqueness. \square

Corollary 1.3. *Let $\alpha \in (0, 1]$ and $s, r \in [0, \infty)$ be fixed. For any $t \in [0, \infty)$ and any positive integer n ,*

$$\frac{1}{n!} \left(\frac{t^\alpha - r^\alpha}{\alpha} \right)^n = \sum_{k=0}^n \frac{1}{k!(n-k)!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^k \left(\frac{s^\alpha - r^\alpha}{\alpha} \right)^{n-k}.$$

Proof. This follows immediately from the theorem if we take $f(t) = \frac{1}{n!} \left(\frac{t^\alpha - r^\alpha}{\alpha} \right)^n$ in Taylor's formula. It can also be shown directly. \square

2. STEFFENSEN INEQUALITY

We begin this section with a definition of α -fractional integrability. The results in this and subsequent sections differ from those in [8].

Definition 2.1. Let $\alpha \in (0, 1]$ and $0 \leq a < b$. A function $f : [a, b] \rightarrow \mathbb{R}$ is α -fractional integrable on $[a, b]$ if the integral

$$\int_a^b f(t) d_\alpha t := \int_a^b f(t) t^{\alpha-1} dt$$

exists and is finite.

Lemma 2.1. Let $\alpha \in (0, 1]$ and $a, b \in \mathbb{R}$ with $0 \leq a < b$. Let $g : [a, b] \rightarrow [0, 1]$ be an α -fractional integrable function on $[a, b]$, and define

$$(2.1) \quad \ell := \frac{\alpha(b-a)}{b^\alpha - a^\alpha} \int_a^b g(t) d_\alpha t \in [0, b-a].$$

Then

$$(2.2) \quad \int_{b-\ell}^b 1 d_\alpha t \leq \int_a^b g(t) d_\alpha t \leq \int_a^{a+\ell} 1 d_\alpha t.$$

Proof. Since $g(t) \in [0, 1]$ for all $t \in [a, b]$, ℓ given in (2.1) satisfies

$$0 \leq \ell = \frac{\alpha(b-a)}{b^\alpha - a^\alpha} \int_a^b g(t) d_\alpha t \leq \frac{\alpha(b-a)}{b^\alpha - a^\alpha} \int_a^b 1 d_\alpha t = \frac{\alpha(b-a)}{b^\alpha - a^\alpha} \frac{b^\alpha - a^\alpha}{\alpha} = b-a.$$

As $\alpha \in (0, 1]$ we have that $t^{\alpha-1}$ is a decreasing function on $[a, b]$, or $(a, b]$ if $a = 0$. Thus using the fact that $d_\alpha t = t^{\alpha-1} dt$ we have the following inequalities, which are average values, namely

$$\frac{1}{\ell} \int_{b-\ell}^b 1 d_\alpha t \leq \frac{1}{b-a} \int_a^b 1 d_\alpha t \leq \frac{1}{\ell} \int_a^{a+\ell} 1 d_\alpha t.$$

This implies that

$$\int_{b-\ell}^b 1 d_\alpha t \leq \frac{\ell}{b-a} \int_a^b 1 d_\alpha t \leq \int_a^{a+\ell} 1 d_\alpha t,$$

which leads to (2.2) via (2.1). \square

Theorem 2.2 (Fractional Steffensen Inequality). Let $\alpha \in (0, 1]$ and $a, b \in \mathbb{R}$ with $0 \leq a < b$. Let $f : [a, b] \rightarrow [0, \infty)$ and $g : [a, b] \rightarrow [0, 1]$ be α -fractional integrable functions on $[a, b]$, with f decreasing. Then

$$(2.3) \quad \int_{b-\ell}^b f(t) d_\alpha t \leq \int_a^b f(t) g(t) d_\alpha t \leq \int_a^{a+\ell} f(t) d_\alpha t,$$

where ℓ is given by (2.1).

Proof. We will prove only the case in (2.3) for the left inequality; the proof for the right inequality is similar, and relies on (2.2).

By the definition of ℓ in (2.1) and the conditions on g , we know that (2.2) holds. After subtracting within the left inequality of (2.3), we see that

$$\begin{aligned}
\int_a^b f(t)g(t)d_\alpha t & - \int_{b-\ell}^b f(t)d_\alpha t \\
& = \int_a^{b-\ell} f(t)g(t)d_\alpha t + \int_{b-\ell}^b f(t)g(t)d_\alpha t - \int_{b-\ell}^b f(t)d_\alpha t \\
& = \int_a^{b-\ell} f(t)g(t)d_\alpha t - \int_{b-\ell}^b f(t)(1-g(t))d_\alpha t \\
& \geq \int_a^{b-\ell} f(t)g(t)d_\alpha t - f(b-\ell) \int_{b-\ell}^b (1-g(t))d_\alpha t \\
& = \int_a^{b-\ell} f(t)g(t)d_\alpha t - f(b-\ell) \left[\int_{b-\ell}^b 1 d_\alpha t - \int_{b-\ell}^b g(t)d_\alpha t \right] \\
& \stackrel{(2.2)}{\geq} \int_a^{b-\ell} f(t)g(t)d_\alpha t - f(b-\ell) \int_a^{b-\ell} g(t)d_\alpha t \\
& = \int_a^{b-\ell} (f(t) - f(b-\ell))g(t)d_\alpha t \geq 0,
\end{aligned}$$

since f is decreasing and g is nonnegative. Therefore the left-hand side of (2.3) holds. \square

Remark 2.1. The requirement in Steffensen's Theorem 2.2 that f be non-negative is essential. For example, let $a = 0$, $b = 1$, $\alpha = 1/2 \equiv g$, and $f \equiv -1$. Then $\ell = 1/2$, and

$$\int_{b-\ell}^b f(t)d_\alpha t = -2 + \sqrt{2} > -1 = \int_a^b f(t)g(t)d_\alpha t,$$

a contradiction of the left-hand side of (2.3).

3. TAYLOR REMAINDER

Let $\alpha \in (0, 1]$ and suppose f is $n + 1$ times α -fractional differentiable on $[0, \infty)$. Using Taylor's Theorem, Theorem 1.2, we define the remainder function by

$$R_{-1,f}(\cdot, s) := f(s),$$

and for $n > -1$,

$$\begin{aligned}
 (3.1) \quad R_{n,f}(t, s) &:= f(s) - \sum_{k=0}^n \frac{D_\alpha^k f(t)}{k!} \left(\frac{s^\alpha - t^\alpha}{\alpha} \right)^k \\
 &= \frac{1}{n!} \int_t^s \left(\frac{s^\alpha - \tau^\alpha}{\alpha} \right)^n D_\alpha^{n+1} f(\tau) d_\alpha \tau.
 \end{aligned}$$

Lemma 3.1. *Let $\alpha \in (0, 1]$. The following identity involving α -fractional Taylor's remainder holds:*

$$\int_a^b \frac{D_\alpha^{n+1} f(s)}{(n+1)!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^{n+1} d_\alpha s = \int_a^t R_{n,f}(a, s) d_\alpha s + \int_t^b R_{n,f}(b, s) d_\alpha s.$$

Proof. We proceed by mathematical induction on n . For $n = -1$,

$$\int_a^b D_\alpha^0 f(s) d_\alpha s = \int_a^b f(s) d_\alpha s = \int_a^t f(s) d_\alpha s + \int_t^b f(s) d_\alpha s.$$

Assume the result holds for $n = k - 1$:

$$\int_a^b \frac{D_\alpha^k f(s)}{k!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^k d_\alpha s = \int_a^t R_{k-1,f}(a, s) d_\alpha s + \int_t^b R_{k-1,f}(b, s) d_\alpha s.$$

Let $n = k$. Using integration by parts, we have

$$\begin{aligned}
 \int_a^b \frac{D_\alpha^{k+1} f(s)}{(k+1)!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^{k+1} d_\alpha s &= \frac{D_\alpha^k f(b)}{(k+1)!} \left(\frac{t^\alpha - b^\alpha}{\alpha} \right)^{k+1} \\
 &\quad - \frac{D_\alpha^k f(a)}{(k+1)!} \left(\frac{t^\alpha - a^\alpha}{\alpha} \right)^{k+1} \\
 &\quad + \int_a^b \frac{D_\alpha^k f(s)}{k!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^k d_\alpha s.
 \end{aligned}$$

By the induction assumption,

$$\begin{aligned}
\int_a^b \frac{D_\alpha^{k+1} f(s)}{(k+1)!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^{k+1} d_\alpha s &= \int_a^t R_{k-1,f}(a, s) d_\alpha s + \int_t^b R_{k-1,f}(b, s) d_\alpha s \\
&\quad + \frac{D_\alpha^k f(b)}{(k+1)!} \left(\frac{t^\alpha - b^\alpha}{\alpha} \right)^{k+1} \\
&\quad - \frac{D_\alpha^k f(a)}{(k+1)!} \left(\frac{t^\alpha - a^\alpha}{\alpha} \right)^{k+1} \\
&= \int_a^t R_{k-1,f}(a, s) d_\alpha s + \int_t^b R_{k-1,f}(b, s) d_\alpha s \\
&\quad + \frac{D_\alpha^k f(b)}{k!} \int_b^t \left(\frac{s^\alpha - b^\alpha}{\alpha} \right)^k d_\alpha s \\
&\quad - \frac{D_\alpha^k f(a)}{k!} \int_a^t \left(\frac{s^\alpha - a^\alpha}{\alpha} \right)^k d_\alpha s \\
&= \int_a^t \left[R_{k-1,f}(a, s) - \frac{D_\alpha^k f(a)}{k!} \left(\frac{s^\alpha - a^\alpha}{\alpha} \right)^k \right] d_\alpha s \\
&\quad + \int_t^b \left[R_{k-1,f}(b, s) - \frac{D_\alpha^k f(b)}{k!} \left(\frac{s^\alpha - b^\alpha}{\alpha} \right)^k \right] d_\alpha s \\
&= \int_a^t R_{k,f}(a, s) d_\alpha s + \int_t^b R_{k,f}(b, s) d_\alpha s.
\end{aligned}$$

This completes the proof. \square

Corollary 3.2. *Let $\alpha \in (0, 1]$. For $n \geq -1$,*

$$\begin{aligned}
\int_a^b \frac{D_\alpha^{n+1} f(s)}{(n+1)!} \left(\frac{a^\alpha - s^\alpha}{\alpha} \right)^{n+1} d_\alpha s &= \int_a^b R_{n,f}(b, s) d_\alpha s, \\
\int_a^b \frac{D_\alpha^{n+1} f(s)}{(n+1)!} \left(\frac{b^\alpha - s^\alpha}{\alpha} \right)^{n+1} d_\alpha s &= \int_a^b R_{n,f}(a, s) d_\alpha s.
\end{aligned}$$

4. APPLICATIONS OF THE STEFFENSEN INEQUALITY

Let $\alpha \in (0, 1]$. In the following we adapt to the α -fractional setting some results from [5] by applying the fractional Steffensen inequality, Theorem 2.2.

Theorem 4.1. *Let $\alpha \in (0, 1]$ and $f : [a, b] \rightarrow \mathbb{R}$ be an $n + 1$ times α -fractional differentiable function such that $D_\alpha^{n+1} f$ is increasing and $D_\alpha^n f$ is decreasing on $[a, b]$. If*

$$\ell := \frac{b - a}{n + 2},$$

then

$$\begin{aligned} D_\alpha^n f(a + \ell) - D_\alpha^n f(a) &\leq (n+1)! \left(\frac{\alpha}{b^\alpha - a^\alpha} \right)^{n+1} \int_a^b R_{n,f}(a, s) d_\alpha s \\ &\leq D_\alpha^n f(b) - D_\alpha^n f(b - \ell). \end{aligned}$$

Proof. Let $F := -D_\alpha^{n+1} f$. Because $D_\alpha^n f$ is decreasing, $D_\alpha^{n+1} f \leq 0$, so that $F \geq 0$ and decreasing on $[a, b]$. Define

$$g(t) := \left(\frac{b^\alpha - t^\alpha}{b^\alpha - a^\alpha} \right)^{n+1} \in [0, 1], \quad t \in [a, b], \quad n \geq -1.$$

Note that F, g satisfy the assumptions of Steffensen's inequality, Theorem 2.2; using (2.1),

$$\ell = \frac{\alpha(b-a)}{b^\alpha - a^\alpha} \int_a^b g(t) d_\alpha t = \frac{b-a}{n+2},$$

and

$$-\int_{b-\ell}^b D_\alpha^{n+1} f(t) d_\alpha t \leq -\int_a^b D_\alpha^{n+1} f(t) \left(\frac{b^\alpha - t^\alpha}{b^\alpha - a^\alpha} \right)^{n+1} d_\alpha t \leq -\int_a^{a+\ell} D_\alpha^{n+1} f(t) d_\alpha t.$$

By Corollary 3.2 this simplifies to

$$D_\alpha^n f(t)|_{t=a}^{a+\ell} \leq (n+1)! \left(\frac{\alpha}{b^\alpha - a^\alpha} \right)^{n+1} \int_a^b R_{n,f}(a, t) d_\alpha t \leq D_\alpha^n f(t)|_{t=b-\ell}^b.$$

This completes the proof. \square

The following corollary is the first Hermite-Hadamard inequality, derived from Theorem 4.1 with $n = 0$.

Corollary 4.2 (Hermite-Hadamard Inequality I). *Let $\alpha \in (0, 1]$ and $f : [a, b] \rightarrow \mathbb{R}$ be an α -fractional differentiable function such that $D_\alpha f$ is increasing and f is decreasing on $[a, b]$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(s) d_\alpha s \leq f(b) + f(a) - f\left(\frac{a+b}{2}\right).$$

Theorem 4.3. *Let $\alpha \in (0, 1]$ and $f : [a, b] \rightarrow \mathbb{R}$ be an $n+1$ times α -fractional differentiable function such that*

$$m \leq D_\alpha^{n+1} f \leq M$$

on $[a, b]$ for some real numbers $m < M$. Then

$$\begin{aligned} (4.1) \quad &\frac{m}{(n+2)!} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{n+2} + \frac{M-m}{(n+2)!} \left(\frac{b^\alpha - (b-\ell)^\alpha}{\alpha} \right)^{n+2} \leq \int_a^b R_{n,f}(a, t) d_\alpha t \\ &\leq \frac{M}{(n+2)!} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{n+2} + \frac{m-M}{(n+2)!} \left(\frac{b^\alpha - (a+\ell)^\alpha}{\alpha} \right)^{n+2}, \end{aligned}$$

where ℓ is given by

$$\ell = \frac{\alpha(b-a)}{(b^\alpha - a^\alpha)(M-m)} \left(D_\alpha^n f(b) - D_\alpha^n f(a) - m \left(\frac{b^\alpha - a^\alpha}{\alpha} \right) \right)$$

Proof. Let

$$\begin{aligned} F(t) &:= \frac{1}{(n+1)!} \left(\frac{b^\alpha - t^\alpha}{\alpha} \right)^{n+1}, \\ k(t) &:= \frac{1}{M-m} \left(f(t) - \frac{m}{(n+1)!} \left(\frac{t^\alpha - a^\alpha}{\alpha} \right)^{n+1} \right), \\ G(t) &:= D_\alpha^{n+1} k(t) = \frac{1}{M-m} (D_\alpha^{n+1} f(t) - m) \in [0, 1]. \end{aligned}$$

Observe that F is nonnegative and decreasing, and

$$\int_a^b G(t) d_\alpha t = \frac{1}{M-m} \left(D_\alpha^n f(b) - D_\alpha^n f(a) - m \left(\frac{b^\alpha - a^\alpha}{\alpha} \right) \right).$$

Since F, G satisfy the hypotheses of Theorem 2.2, we compute the various integrals given in (2.3), after using (2.1) to set

$$\ell = \frac{\alpha(b-a)}{b^\alpha - a^\alpha} \int_a^b G(t) d_\alpha t.$$

We have

$$\int_{b-\ell}^b F(t) d_\alpha t = \int_{b-\ell}^b \frac{1}{(n+1)!} \left(\frac{b^\alpha - t^\alpha}{\alpha} \right)^{n+1} d_\alpha t = \frac{1}{(n+2)!} \left(\frac{b^\alpha - (b-\ell)^\alpha}{\alpha} \right)^{n+2},$$

and

$$\int_a^{a+\ell} F(t) d_\alpha t = \frac{1}{(n+2)!} \left[\left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{n+2} - \left(\frac{b^\alpha - (a+\ell)^\alpha}{\alpha} \right)^{n+2} \right].$$

Moreover, using Corollary 3.2, we have

$$\begin{aligned} \int_a^b F(t) G(t) d_\alpha t &= \frac{1}{(M-m)(n+1)!} \int_a^b \left(\frac{b^\alpha - t^\alpha}{\alpha} \right)^{n+1} (D_\alpha^{n+1} f(t) - m) d_\alpha t \\ &= \frac{1}{M-m} \int_a^b R_{n,f}(a, t) d_\alpha t - \frac{m}{(M-m)(n+2)!} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{n+2}. \end{aligned}$$

Using Steffensen's inequality (2.3) and some rearranging, we obtain (4.1). \square

Corollary 4.4. *Let $\alpha \in (0, 1]$ and $f : [a, b] \rightarrow \mathbb{R}$ be an α -fractional differentiable function such that*

$$m \leq D_\alpha f \leq M$$

on $[a, b]$ for some real numbers $m < M$. Then

$$(4.2) \quad \begin{aligned} \frac{m}{2} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^2 + \frac{M-m}{2} \left(\frac{b^\alpha - (b-\ell)^\alpha}{\alpha} \right)^2 &\leq \int_a^b f(t) d_\alpha t - f(a) \left(\frac{b^\alpha - a^\alpha}{\alpha} \right) \\ &\leq \frac{M}{2} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^2 + \frac{m-M}{2} \left(\frac{b^\alpha - (a+\ell)^\alpha}{\alpha} \right)^2, \end{aligned}$$

where ℓ is given by

$$\ell = \frac{\alpha(b-a)}{(b^\alpha - a^\alpha)(M-m)} \left(f(b) - f(a) - m \left(\frac{b^\alpha - a^\alpha}{\alpha} \right) \right)$$

Proof. Use the previous theorem with $n = 0$, and Corollary 3.2. \square

5. APPLICATIONS OF THE ČEBYŠEV INEQUALITY

Let $\alpha \in (0, 1]$. We begin with Čebyšev's inequality for α -fractional integrals, then apply it to obtain a Hermite-Hadamard-type inequality.

Theorem 5.1 (Čebyšev Inequality). *Let f and g be both increasing or both decreasing in $[a, b]$, and let $\alpha \in (0, 1]$. Then*

$$\int_a^b f(t)g(t) d_\alpha t \geq \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \int_a^b g(t) d_\alpha t.$$

If one of the functions is increasing and the other is decreasing, then the above inequality is reversed.

Proof. The proof is very similar to the classical case with $\alpha = 1$. \square

The following is an application of Čebyšev's inequality, which extends a similar result in [5] for q -calculus to this α -fractional case.

Theorem 5.2. *Let $\alpha \in (0, 1]$. Assume that $D_\alpha^{n+1}f$ is monotonic on $[a, b]$. If $D_\alpha^{n+1}f$ is increasing, then*

$$\begin{aligned} 0 &\geq \int_a^b R_{n,f}(a, t) d_\alpha t - \left(\frac{D_\alpha^n f(b) - D_\alpha^n f(a)}{(n+2)!} \right) \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{n+1} \\ &\geq \left(\frac{D_\alpha^{n+1}f(a) - D_\alpha^{n+1}f(b)}{(n+2)!} \right) \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{n+2}. \end{aligned}$$

If $D_\alpha^{n+1}f$ is decreasing, then the inequalities are reversed.

Proof. The situation where $D_\alpha^{n+1}f$ is decreasing is analogous to that of $D_\alpha^{n+1}f$ increasing. Thus, assume $D_\alpha^{n+1}f$ is increasing and set

$$F(t) := D_\alpha^{n+1}f(t), \quad G(t) := \frac{1}{(n+1)!} \left(\frac{b^\alpha - t^\alpha}{\alpha} \right)^{n+1}.$$

Then F is increasing by assumption, and G is decreasing, so that by Čebyšev's inequality,

$$\int_a^b F(t)G(t)d_\alpha t \leq \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b F(t)d_\alpha t \int_a^b G(t)d_\alpha t.$$

By Corollary 3.2,

$$\int_a^b F(t)G(t)d_\alpha t = \int_a^b \frac{D_\alpha^{n+1}f(t)}{(n+1)!} \left(\frac{b^\alpha - t^\alpha}{\alpha} \right)^{n+1} d_\alpha t = \int_a^b R_{n,f}(a, t)d_\alpha t.$$

We also have

$$\int_a^b F(t)d_\alpha t = D_\alpha^n f(b) - D_\alpha^n f(a), \quad \int_a^b G(t)d_\alpha t = \frac{1}{(n+2)!} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{n+2}.$$

Thus Čebyšev's inequality implies

$$\int_a^b R_{n,f}(a, t)d_\alpha t \leq \frac{\alpha}{b^\alpha - a^\alpha} (D_\alpha^n f(b) - D_\alpha^n f(a)) \frac{1}{(n+2)!} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{n+2},$$

which subtracts to the left side of the inequality. Since $D_\alpha^{n+1}f$ is increasing on $[a, b]$,

$$\begin{aligned} \frac{D_\alpha^{n+1}f(a)}{(n+2)!} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{n+2} &\leq \left(\frac{D_\alpha^n f(b) - D_\alpha^n f(a)}{(n+2)!} \right) \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{n+1} \\ &\leq \frac{D_\alpha^{n+1}f(b)}{(n+2)!} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{n+2}, \end{aligned}$$

and we have

$$\begin{aligned} \int_a^b R_{n,f}(a, t)d_\alpha t - \left(\frac{D_\alpha^n f(b) - D_\alpha^n f(a)}{(n+2)!} \right) \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{n+1} \\ \geq \int_a^b R_{n,f}(a, t)d_\alpha t - \frac{D_\alpha^{n+1}f(b)}{(n+2)!} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{n+2}. \end{aligned}$$

Now Corollary 3.2 and $D_\alpha^{n+1}f$ is increasing imply that

$$\begin{aligned} \int_a^b \frac{D_\alpha^{n+1}f(b)}{(n+1)!} \left(\frac{b^\alpha - t^\alpha}{\alpha} \right)^{n+1} d_\alpha t &\geq \int_a^b R_{n,f}(a, t)d_\alpha t \\ &\geq \int_a^b \frac{D_\alpha^{n+1}f(a)}{(n+1)!} \left(\frac{b^\alpha - t^\alpha}{\alpha} \right)^{n+1} d_\alpha t, \end{aligned}$$

which simplifies to

$$\frac{D_\alpha^{n+1}f(b)}{(n+2)!} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{n+2} \geq \int_a^b R_{n,f}(a, t)d_\alpha t \geq \frac{D_\alpha^{n+1}f(a)}{(n+2)!} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{n+2}.$$

This, together with the earlier lines give the right side of the inequality. \square

Compare the following corollary with Corollary 4.2.

Corollary 5.3 (Hermite-Hadamard Inequality II). *Let $\alpha \in (0, 1]$. If $D_\alpha f$ is increasing on $[a, b]$, then*

$$(5.1) \quad \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \leq \frac{f(b) + f(a)}{2}.$$

If $D_\alpha f$ is decreasing on $[a, b]$, then the inequalities are reversed.

6. OSTROWSKI INEQUALITY

In this section we prove Ostrowski's α -fractional inequality using a Montgomery identity. For more on Ostrowski's inequalities, see [1] and the references therein.

Lemma 6.1 (Montgomery Identity). *Let $a, b, s, t \in \mathbb{R}$ with $0 \leq a < b$, and let $f : [a, b] \rightarrow \mathbb{R}$ be α -fractional differentiable for $\alpha \in (0, 1]$. Then*

$$(6.1) \quad f(t) = \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(s) d_\alpha s + \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b p(t, s) D_\alpha f(s) d_\alpha s$$

where

$$(6.2) \quad p(t, s) := \begin{cases} \frac{s^\alpha - a^\alpha}{\alpha} & : a \leq s < t, \\ \frac{s^\alpha - b^\alpha}{\alpha} & : t \leq s \leq b. \end{cases}$$

Proof. Integrating by parts, we have

$$\int_a^t \left(\frac{s^\alpha - a^\alpha}{\alpha} \right) D_\alpha f(s) d_\alpha s = \frac{t^\alpha - a^\alpha}{\alpha} f(t) - \int_a^t f(s) d_\alpha s$$

and

$$\int_t^b \left(\frac{s^\alpha - b^\alpha}{\alpha} \right) D_\alpha f(s) d_\alpha s = \frac{b^\alpha - t^\alpha}{\alpha} f(t) - \int_t^b f(s) d_\alpha s.$$

Adding and solving for f yields the result. \square

Theorem 6.2 (Ostrowski Inequality). *Let $a, b, s, t \in \mathbb{R}$ with $0 \leq a < b$, and let $f : [a, b] \rightarrow \mathbb{R}$ be α -fractional differentiable for $\alpha \in (0, 1]$. Then*

$$(6.3) \quad \left| f(t) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right| \leq \frac{M}{2\alpha(b^\alpha - a^\alpha)} [(t^\alpha - a^\alpha)^2 + (b^\alpha - t^\alpha)^2],$$

where

$$M := \sup_{t \in (a, b)} |D_\alpha f(t)|.$$

This inequality is sharp in the sense that the right-hand side of (6.3) cannot be replaced by a smaller one.

Proof. Using Lemma 6.1 with $p(t, s)$ defined in 6.2 we see that

$$\begin{aligned}
\left| f(t) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(s) d_\alpha s \right| &= \left| \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b p(t, s) D_\alpha f(s) d_\alpha s \right| \\
&\leq \frac{M\alpha}{b^\alpha - a^\alpha} \left(\int_a^t \left| \frac{s^\alpha - a^\alpha}{\alpha} \right| d_\alpha s + \int_t^b \left| \frac{s^\alpha - b^\alpha}{\alpha} \right| d_\alpha s \right) \\
&= \frac{M\alpha}{b^\alpha - a^\alpha} \left(\int_a^t \left(\frac{s^\alpha - a^\alpha}{\alpha} \right) d_\alpha s + \int_t^b \left(\frac{b^\alpha - s^\alpha}{\alpha} \right) d_\alpha s \right) \\
&= \frac{M\alpha}{b^\alpha - a^\alpha} \left(\frac{1}{2} \left(\frac{s^\alpha - a^\alpha}{\alpha} \right)^2 \Big|_a^t - \frac{1}{2} \left(\frac{b^\alpha - s^\alpha}{\alpha} \right)^2 \Big|_t^b \right) \\
&= \frac{M}{2\alpha(b^\alpha - a^\alpha)} [(t^\alpha - a^\alpha)^2 + (b^\alpha - t^\alpha)^2].
\end{aligned}$$

Now $p(t, a) = 0$, so the smallest value attaining the supremum in M is greater than a . To prove the sharpness of this inequality, let $f(t) = t^\alpha/\alpha$, $a = t_1$, $b = t_2 = t$. It follows that $D_\alpha f(t) = 1$ and $M = 1$. Examining the right-hand side of (6.3) we get

$$\frac{M}{2\alpha(b^\alpha - a^\alpha)} [(t^\alpha - a^\alpha)^2 + (b^\alpha - t^\alpha)^2] = \frac{(t_2^\alpha - t_1^\alpha)^2}{2\alpha(t_2^\alpha - t_1^\alpha)} = \frac{t_2^\alpha - t_1^\alpha}{2\alpha}.$$

Starting with the left-hand side of (6.3), we have

$$\begin{aligned}
\left| f(t) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right| &= \left| \frac{t^\alpha}{\alpha} - \frac{\alpha}{t_2^\alpha - t_1^\alpha} \int_{t_1}^{t_2} \frac{t^\alpha}{\alpha} d_\alpha t \right| \\
&= \left| \frac{t^\alpha}{\alpha} - \left(\frac{\alpha}{t_2^\alpha - t_1^\alpha} \right) \left(\frac{t^{2\alpha}}{2\alpha^2} \right) \Big|_{t_1}^{t_2} \right| \\
&= \left| \frac{t^\alpha}{\alpha} - \left(\frac{1}{t_2^\alpha - t_1^\alpha} \right) \left(\frac{t_2^{2\alpha} - t_1^{2\alpha}}{2\alpha} \right) \right| \\
&= \left| \frac{t^\alpha}{\alpha} - \left(\frac{t_2^\alpha + t_1^\alpha}{2\alpha} \right) \right| \\
&= \frac{t_2^\alpha - t_1^\alpha}{2\alpha}.
\end{aligned}$$

Therefore by the squeeze theorem the sharpness of Ostrowski's inequality is shown. \square

7. GRÜSS INEQUALITY

In this section we prove the Grüss inequality, which relies on Jensen's inequality. Our approach is similar to that taken by [2].

Theorem 7.1 (Jensen Inequality). *Let $\alpha \in (0, 1]$ and $a, b, x, y \in [0, \infty)$. If $w : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow (x, y)$ are nonnegative, continuous functions with $\int_a^b w(t) d_\alpha t > 0$, and $F : (x, y) \rightarrow \mathbb{R}$ is continuous and convex, then*

$$F\left(\frac{\int_a^b w(t)g(t)d_\alpha t}{\int_a^b w(t)d_\alpha t}\right) \leq \frac{\int_a^b w(t)F(g(t))d_\alpha t}{\int_a^b w(t)d_\alpha t}.$$

Proof. The proof is the same as those found in Bohner and Peterson [3, Theorem 6.17] and Rudin [9, Theorem 3.3] and thus is omitted. \square

Theorem 7.2 (Grüss Inequality). *Let $a, b, s \in [0, \infty)$, and let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous functions. Then for $\alpha \in (0, 1]$ and*

$$(7.1) \quad m_1 \leq f(t) \leq M_1, \quad m_2 \leq g(t) \leq M_2,$$

we have

$$\left| \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t)g(t)d_\alpha t - \left(\frac{\alpha}{b^\alpha - a^\alpha} \right)^2 \int_a^b f(t)d_\alpha t \int_a^b g(t)d_\alpha t \right| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2).$$

Proof. Initially we consider an easier case, namely where $f = g$ and

$$\frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t)d_\alpha t = 0.$$

If we define

$$v(t) := \frac{f(t) - m_1}{M_1 - m_1} \in [0, 1],$$

then $f(t) = m_1 + (M_1 - m_1)v(t)$. Since

$$\int_a^b v^2(t)d_\alpha t \leq \int_a^b v(t)d_\alpha t = \frac{-m_1(b^\alpha - a^\alpha)}{\alpha(M_1 - m_1)},$$

we have

$$\begin{aligned} I(f, f) &:= \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f^2(t)d_\alpha t - \left(\frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t)d_\alpha t \right)^2 \\ &= \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b [m_1 + (M_1 - m_1)v(t)]^2(t)d_\alpha t \\ &\leq -m_1 M_1 = \frac{1}{4} [(M_1 - m_1)^2 - (M_1 + m_1)^2] \\ &\leq \frac{1}{4}(M_1 - m_1)^2. \end{aligned}$$

Now consider the case

$$r := \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \neq 0,$$

where $r \in \mathbb{R}$. If we take $h(t) := f(t) - r$, then $h(t) \in [m_1 - r, M_1 - r]$ and

$$\frac{\alpha}{b^\alpha - a^\alpha} \int_a^b h(t) d_\alpha t = \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b (f(t) - r) d_\alpha t = r - \frac{r\alpha}{b^\alpha - a^\alpha} \int_a^b d_\alpha t = 0.$$

Consequently h satisfies the earlier assumptions and so

$$I(h, h) \leq \frac{1}{4} [M_1 - r - (m_1 - r)]^2 = \frac{1}{4} (M_1 - m_1)^2.$$

Additionally we have

$$I(h, h) = \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b (f(t) - r)^2 d_\alpha t = -r^2 + \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f^2(t) d_\alpha t = I(f, f).$$

As a result,

$$I(f, f) = I(h, h) \leq \frac{1}{4} (M_1 - m_1)^2.$$

Let us now turn to the case involving general functions f and g under assumptions (7.1). Using

$$I(f, g) := \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t)g(t) d_\alpha t - \left(\frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right)^2 \int_a^b g(t) d_\alpha t$$

and the earlier cases, one can easily finish the proof as in the case with $\alpha = 1$. See [2] for complete details to mimic. \square

Corollary 7.3. *Let $\alpha \in (0, 1]$, $a, b, s, t \in [0, \infty)$ and $f : [a, b] \rightarrow \mathbb{R}$ be α -fractional differentiable. If $D_\alpha f$ is continuous and*

$$m \leq D_\alpha f(t) \leq M, \quad t \in [a, b],$$

then

$$(7.2) \quad \left| f(t) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(s) d_\alpha s - \left[\frac{2t^\alpha - a^\alpha - b^\alpha}{2(b^\alpha - a^\alpha)} \right] [f(b) - f(a)] \right| \leq \frac{1}{4} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right) (M - m).$$

for all $t \in [a, b]$.

Proof. Using Lemma 6.1 Montgomery's identity we have

$$(7.3) \quad f(t) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(s) d_\alpha s = \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b p(t, s) D_\alpha f(s) d_\alpha s$$

for all $t \in [a, b]$, where $p(t, s)$ is given in (6.2). Now for all $t, s \in [a, b]$, we see that

$$\frac{t^\alpha - b^\alpha}{\alpha} \leq p(t, s) \leq \frac{t^\alpha - a^\alpha}{\alpha}.$$

Applying Theorem 7.2 Grüss' inequality to the mappings $p(t, \cdot)$ and $D_\alpha f$, we obtain

$$(7.4) \quad \left| \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b p(t, s) D_\alpha f(s) d_\alpha s - \left(\frac{\alpha}{b^\alpha - a^\alpha} \right)^2 \int_a^b p(t, s) d_\alpha s \int_a^b D_\alpha f(s) d_\alpha s \right| \\ \leq \frac{1}{4} \left(\frac{t^\alpha - a^\alpha}{\alpha} - \frac{t^\alpha - b^\alpha}{\alpha} \right) (M - m) = \frac{1}{4} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right) (M - m).$$

Computing the integrals involved, we obtain

$$\left(\frac{\alpha}{b^\alpha - a^\alpha} \right)^2 \int_a^b p(t, s) d_\alpha s = \frac{2t^\alpha - a^\alpha - b^\alpha}{2(b^\alpha - a^\alpha)}$$

and

$$\int_a^b D_\alpha f(s) d_\alpha s = f(b) - f(a),$$

so that (7.2) holds, after using (7.3) and (7.4). \square

Compare the following corollary with Corollary 4.2 and Corollary 5.3.

Corollary 7.4 (Hermite-Hadamard III). *Let $\alpha \in (0, 1]$, $a, b, s, t \in [0, \infty)$ and $f : [a, b] \rightarrow \mathbb{R}$ be α -fractional differentiable. If $D_\alpha f$ is continuous and*

$$m \leq D_\alpha f(t) \leq M, \quad t \in [a, b],$$

then

$$\left| \frac{f(b) + f(a)}{2} - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(s) d_\alpha s \right| \leq \frac{1}{4} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right) (M - m).$$

for all $t \in [a, b]$.

Proof. Take $t = b$ in the previous corollary. \square

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